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An Iterative Scheme for the N-Competing Species Problem*

ANTONIO TINEO

Universidad de Los Andes, Facultad de Ciencias, Departamento de Matematicas, 5101 Merida, Venezuela

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0. INTRODUCTION

In this paper we consider the Lotka–Volterra system

$$u_i' = u_i\left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)\, u_j\right], \qquad 1 \leqslant i \leqslant n, \tag{0.1}$$

where $a_i, b_{ij}\colon \mathbb{R} \to \mathbb{R}$ are positive, continuous, and T-periodic functions, for some fixed $T > 0$.

When a_i, b_{ij} are constants $i, j = 1, ..., n$, the system (0.1) models the competition between n biological species for a common pool of resources.

In this paper we use the iterative scheme

$$u_{i0} \equiv 0, \qquad i = 1, ..., n$$

$$u_{ik}' = u_{ik}\left[a_i - b_{ii}u_{ik} - \sum_{j\in J_i} b_{ij}u_{j,k-1}\right], \qquad k = 1, 2, ...$$

to prove the existence of a coexistence state for (0.1). Here and henceforth, $J_i = \{1, ..., i-1, i+1, ..., n\}$. More precisely, let U_i be the unique positive and T-periodic solution to the logistic equation

$$x' = x[a_i(t) - b_{ii}(t)x].$$

We have:

0.1. THEOREM. *Assume*

$$\int_0^T \left[a_i(t) - \sum_{j\in J_i} b_{ij}(t)\, U_j(t)\right] dt > 0, \qquad 1 \leqslant i \leqslant n. \tag{0.2}$$

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Then, (0.1) *has a T-periodic solution* $u^0 = (u_1^0, ..., u_n^0)$, *such that* $u_i^0 > 0$ *for all i.*

Actually, we shall obtain Theorem 0.1 as a corollary to an existence theorem concerning a competitive system, [8], of the Kolmogorov form

$$u_i' = u_i F_i(t, u_1, ..., u_n), \qquad 1 \leqslant i \leqslant n, \tag{0.3}$$

where $F_i: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, ..., n$, are continuous functions which are T-periodic in the time variable t.

We also study uniqueness and stability of the solution u^0, given by Theorem 0.1. In particular, we shall prove:

0.2. THEOREM. *Suppose that* b_{ij}/a_i *are constant for all* $i, j = 1, ..., n$. *If*

$$\sum_{j \in J_i} b_{ij} a_j / b_{jj} < a_i, \qquad 1 \leqslant i \leqslant n, \tag{0.4}$$

then there exist positive constants $u_1^0, ..., u_n^0$ *such that*

$$u_i(t) \to u_i^0 \qquad as \quad t \to \infty, \qquad i = 1, ..., n,$$

for any positive solution $(u_1, ..., u_n)$ *to* (0.1).

For $n = 2$, Theorem 0.1 was proved in [7, Theorem 5.3] and Theorem 0.2 was proved in [10]. For general n, Theorem 0.1 improves the existence part of the main theorem in [9] and Theorem 0.2 improves a theorem in [5].

Several results in this paper, including Theorem 0.2, remain true if a_i, b_{ij} are merely assumed to be continuous and bounded above and below by positive constants on $\mathbb{R}$. This assumption is reasonable, since few things are truly periodic. See [1] and Section 4 for details.

1. THE LOGISTIC EQUATION

In this section we shall study the following one-dimensional system

$$x' = xF(t, x), \tag{1.1}$$

where $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function that is T-periodic in the time variable t.

In order to apply the usual theorems about ODE, we shall assume that F is locally Lipschitz with respect to the second variable. That is, for each (t_0, x_0) in $\mathbb{R}^2$, there exists a neighborhood N ot his point in $\mathbb{R}^2$ and a positive constant M such that $|F(t, x) - F(t, y)| \leqslant M\,|x - y|$ for all (t, x), (t, y) in N.

1.1. THEOREM. *Suppose that*:

(a) $F(t, x) < F(t, y)$ *if* $0 \leqslant x < y$;

(b) *there exists a positive constant* M *such that* $F(t, M) \leqslant 0$ *for all* t;

(c) $\int_0^T F(t, 0)\, dt > 0$.

Then, Eq. (1.1) *has exactly a T-periodic and positive solution U. In fact,* $U(t) \leqslant M$.

Moreover, the solution u to the initial value problem

$$x' = xF(t, x), \qquad x(t_0) = x_0 > 0 \tag{1.2}$$

is defined on $[t_0, \infty)$ *and*

$$u(t) - U(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Proof. Let u be the solution to (1.2), then $u(t) > 0$ for all t in the domain $\mathrm{Dom}(u)$ of u. On the other hand, it is easy to prove that

$$u(t) \leqslant \max\{u(t_0), M\} \quad \text{in} \quad [t_0, \infty) \cap \mathrm{Dom}(u). \tag{1.3}$$

Thus u is bounded in $[t_0, \infty) \cap \mathrm{Dom}(u)$ and, hence, u is defined in $[t_0, \infty)$.

For $x \geqslant 0$, let $u(t, x)$ be the solution to (1.1) such that $u(t, x) = x$ and define the Poincaré map, P: $[0, \infty) \to [0, \infty)$, by $P(x) = u(T, x)$. We have $P(0) = 0$, $P'(0) = \exp(\int_0^T F(t, 0)\, dt) > 1$, and, by (1.3), $P(M) \leqslant M$. Hence, P has a fixed point $z \in (0, M]$ and then $U(t) := u(t, z)$ is a T-periodic and positive solution to (1.1).

Let V, W be positive and T-periodic solutions to (1.1) and define $h = (V/W) - 1$. Then, h is T-periodic and

$$h' = VW^{-1}[F(t, V) - F(t, W)]. \tag{1.4}$$

If $V \not\equiv W$, we can assume that $V < W$ and by (1.4), we get $h' > 0$. In particular, h is not periodic and this contradiction proves that (1.1) has exactly one T-periodic and positive solution. Equivalently, z is unique fixed point of P in $(0, \infty)$.

Let $u(t) = u(t, x_0)$ be the solution to (1.2). Without loss of generality, we can assume that $t_0 = 0$. Since $u(t + T)$ is also a solution to (1.1), $\{u(nT) : n = 1, 2. ...\}$ is a positive monotonic sequence of $\mathbb{R}$. In particular, $u(nT) \to q$, for some $q \geqslant 0$.

CLAIM. $q > 0$.

Proof. Assume $q=0$, then $\{u(nT)\}$ is strictly decreasing and $u(t)\to 0$ as $t\to\infty$. Now ,let us fix $\varepsilon>0$ such that $\int_0^T F(t,\varepsilon)\,dt>0$ and choose $t_1>t_0$ such that $u(t)\leqslant\varepsilon$ for all $t\geqslant t_1$. Integrating the relation $u'(t)/u(t)=F(t,u(t))$ in the interval $[nT,(n+1)T]$ we get

$$0>\ln\,(u(nT+T)/u(nT))=\int_{nT}^{nT+T} F(t,u(t))\,dt\geqslant\int_{nT}^{nT+T} F(t,\varepsilon)\,dt$$

$$=\int_0^T F(t,\varepsilon)\,dt>0$$

for all integers $n\geqslant t_1/T$. This contradiction proves the claim.

From this, q is a positive fixed point of P since $P^n(x_0)=u(nT)\to q$. Consequently, $q=z$ and then $u(t)-U(t)\to 0$ as $t\to\infty$. The proof is complete.

1.2. *Remark.* If $F(t,M)\equiv 0$, then the constant M is a positive T-periodic solution to (1.1) and, hence, the solution U given by Theorem 1.1 is constant $\equiv M$.

1.3. PROPOSITION. *Let $F, G\colon\mathbb{R}^2\to\mathbb{R}$ be continuous functions which are T-periodic with respect to the first variable. Assume that* (1.1) (resp. $x'=xG(t,x)$) *has a T-periodic and positive solution U* (resp. V). *If $F\leqslant G$ and G is strictly decreasing with respect to x, then $U\leqslant V$.*

Proof. Let us define $h=(V/U)-1$. Then, h is periodic and $h'=VU^{-1}[G(t,V)-F(t,U)]\geqslant VU^{-1}[G(t,V)-G(t,U)]$.

Assume now $h(\tau)<0$ for some τ. Then $h'(\tau)>0$ and, hence, $h<0$ in $(-\infty,\tau)$ and $h'<0$ in this interval. This is a contradiction (since h is periodic), and the proof follows easily.

2. An Iterative Scheme

In this section $F_i(t,x)$, $1\leqslant i\leqslant n$, denote real continuous functions, defined on $\mathbb{R}\times\mathbb{R}^n$, which are T-periodic in the time variable t and locally Lipschitz with respect to x. We also assume that:

(H1) $F_i(t,x)>F_i(t,y)$ if $0\leqslant x\leqslant y$, $x\neq y$, and $i=1,\ldots,n$.

Here, $x\leqslant y$ denotes the usual order in $\mathbb{R}^n$.

(H2) There exist positive constants $M_1,\ldots,M_n$ such that

$$F_i(t,0,\ldots,0,M_i,0,\ldots,0)\leqslant 0 \qquad \text{for all } t \text{ in } \mathbb{R} \text{ and } i=1,\ldots,n.$$

From Theorem 1.1, we know that the problem

$$x' = xF_i(t, 0, ..., 0, x, 0, ..., 0), \qquad x(t+T) = x(t), \qquad x > 0$$

has exactly one solution, which we shall denote by U_i.

2.1. THEOREM. *Assume*

$$\int_0^T F_i(t, U_1(t), ..., U_{i-1}(t), 0, U_{i+1}(t), ..., U_n(t))\, dt > 0 \tag{2.1}$$

for $i = 1, ..., n$. *Then, there are T-periodic continuous functions* $\bar{u} = (\bar{u}_1, ..., \bar{u}_n)$, $\underline{u} = (\underline{u}_1, ..., \underline{u}_n)$: $\mathbb{R} \to \mathbb{R}^n$ *with the following properties*:

(a) $\bar{u}_i \geqslant \underline{u}_i > 0$, $i = 1, ..., n$, *and*

$$\begin{aligned} \bar{u}_i' &= \bar{u}_i F_i(t, \underline{u}_1, ..., \underline{u}_{i-1}, \bar{u}_i, \underline{u}_{i+1}, ..., \underline{u}_n) \\ \underline{u}_i' &= \underline{u}_i F_i(t, \bar{u}_1, ..., \bar{u}_{i-1}, \underline{u}_i, \bar{u}_{i+1}, ..., \bar{u}_n). \end{aligned} \tag{2.2}$$

In particular, $(\bar{u}_1, \underline{u}_2)$, $(\underline{u}_1, \bar{u}_2)$ *are solutions to* (0.1) *if* $n = 2$.

(b) $\bar{u} = \underline{u}$ *if* $\bar{u}_i(t_0) = \underline{u}_i(t_0)$ *for some* (i, t_0).

(c) $\underline{u} \leqslant \underline{v}$ *and* $\bar{v} \leqslant \bar{u}$, *for any positive T-periodic solution* $(\bar{v}, \underline{v})$ *to* (2.2). *In particular*, $\underline{u} \leqslant u \leqslant \bar{u}$, *for any T-periodic positive solution* $u = (u_1, ..., u_n)$ *to* (0.3), *since* (u, u) *is a solution to* (2.2).

Proof. Let us define $\bar{u}_{i1} = U_i$. Using Theorem 1.1 and Proposition 1.3 we can prove, inductively, the existence of two sequences $\{\bar{u}_k = (\bar{u}_{1k}, ..., \bar{u}_{nk}) : \mathbb{R} \to \mathbb{R}^n\}$, $\{\underline{u}_k = (\underline{u}_{1k}, ..., \underline{u}_{nk}) : \mathbb{R} \to \mathbb{R}^n\}$ of T-periodic positive functions such that

$$\begin{aligned} \underline{u}_{ik}' &= \underline{u}_{ik} F_i(t, \bar{u}_{1k}, ..., \bar{u}_{i-1,k}, \underline{u}_{ik}, \bar{u}_{i+1,k}, ..., \bar{u}_{nk}) \\ \bar{u}_{i,k+1} &= \bar{u}_{i,k+1} F_i(t, \underline{u}_{1k}, ..., \underline{u}_{i-1,k}, \bar{u}_{i,k+1}, \underline{u}_{i+1,k}, ..., \underline{u}_{nk}) \\ \underline{u}_1 &\leqslant \cdots \leqslant \underline{u}_k \leqslant \bar{u}_k \leqslant \cdots \leqslant \bar{u}_1 \end{aligned} \tag{2.3}$$

for all integers $k \geqslant 1$. Note that $\underline{u}_{i1} \leqslant U_i$ and then (2.1) is satisfied if we replace U_i by $\underline{u}_{i1}$.

In particular, we can define T-periodic positive functions $\bar{u}, \underline{u} : \mathbb{R} \to \mathbb{R}^n$ by

$$\bar{u}(t) = \lim_{k \to \infty} \bar{u}_k(t), \qquad \underline{u}(t) = \lim_{k \to \infty} \underline{u}_k(t).$$

Since $\{\underline{u}_k\}$, $\{\bar{u}_k\}$ are (uniformly) bounded sequences, the same holds for the sequences $\{\underline{u}_k'\}$, $\{\bar{u}_k'\}$, see (2.3). Hence, by Ascoli's theorem and an elementary result about limits and derivatives, we conclude that $(\bar{u}, \underline{u})$ is a solution to (2.2). Thus, the proof of (a) is complete.

If $\bar{u}_i(t_0) = \underline{u}_i(t_0)$ then, $\bar{u}'_i(t_0) = \underline{u}'_i(t_0)$ and the proof of (b) follows from the assumption in (H1).

To prove (c), we remark that $\bar{v}_i$ is a T-periodic and positive solution to the equation

$$x' = xF_i(t, \underline{v}_1(t), ..., \underline{v}_{i-1}(t), x, \underline{v}_{i+1}(t), ..., \underline{v}_n(t))$$

and hence $\bar{v}_i \leqslant U_i$, since $\underline{v}_j > 0$. By induction on k, we get $\bar{v} \leqslant \bar{u}_k$ and $\underline{u}_k \geqslant \underline{v}$ for all k. Thus, the proof is complete.

2.2 *Remark.* Assume $F_i(t, 0, ..., 0, M_i, 0, ..., 0) \equiv 0$ for all i. By Remark 1.2, $U_i \equiv M_i$ and by induction, the sequences $\{\bar{u}_k\}$, $\{\underline{u}_k\}$ in the proof of theorem above are constant. Therefore, the solution $(\bar{u}, \underline{u})$ to (2.2) is constant. In particular, we have proved the following result:

2.3. COROLLARY. *Let $F = (F_1, ..., F_n): \mathbb{R}^n \to \mathbb{R}^n$ be a locally Lipschitz function such that $F(0) > 0$ and $F(x) > F(y)$ if $0 \leqslant x \leqslant y$ and $x \neq y$. Assume further that there are positive constants M_i, $1 \leqslant i \leqslant n$, such that $F_i(M_1, ..., M_{i-1}, 0, M_{i+1}, ..., M_n) > 0$ and $F_i(0, ..., 0, M_i, 0, ..., 0) = 0$. Then, there exists $x > 0$ in $\mathbb{R}^n$ such that $F(x) = 0$.*

Given a point $q = (q_1, ..., q_n)$ in $\mathbb{R}^n$, we denote by $u(t, q)$ the solution to (0.3), satisfying the initial condition $u(0) = q$.

2.4. PROPOSITION. *Assume* (2.1) *and let $\bar{u}$, $\underline{u}$ be as in Theorem* 2.1. *If $n \geqslant 3$ and $\underline{u} < \bar{u}$ and $\underline{u}(0) < q < \bar{u}(0)$, $q \in \mathbb{R}^n$, then $\underline{u}(t) < u(t, q) < \bar{u}(t)$ for all $t > 0$.*

Proof. Let us write $u(t) = u(t, q)$ and assume that there is $t_0 > 0$ such that $\underline{u}(t) < u(t) < \bar{u}(t)$ for $t \in [0, t_0)$ and $u_i(t_0) = \bar{u}_i(t_0)$ (or $\underline{u}_i(t_0) = u_i(t_0)$) for some i. Then,

$$\begin{aligned} &F_i(t_0, \underline{u}_1(t_0), ..., \underline{u}_{i-1}(t_0), \bar{u}_i(t_0), \underline{u}_{i+1}(t_0), ..., \underline{u}_n(t_0)) \\ &\quad \leqslant F_i(t_0, u_1(t_0), ..., u_n(t_0)) \end{aligned}$$

and by (H1), $\underline{u}_j(t_0) = u_j(t_0)$ for all $j \in J_i$.

Let us fix k in J_i. Since $n \geqslant 3$, we can fix j in $\{1, ..., n\}$, such that $j \neq i, k$. On the other hand, $\underline{u}_j < u_j$ in $[0, t_0)$ and $\underline{u}_j(t_0) = u_j(t_0)$, and hence

$$\begin{aligned} &F_j(t_0, \bar{u}_1(t_0), ..., \bar{u}_{j-1}(t_0), \underline{u}_j(t_0), \bar{u}_{j+1}(t_0), ..., \bar{u}_n(t_0)) \\ &\quad \geqslant F_j(t_0, u_1(t_0), ..., u_n(t_0)). \end{aligned}$$

Consequently, $\bar{u}_r(t_0) = u_r(t_0)$ for all $r \neq j$. In particular, $\bar{u}_k(t_0) = u_k(t_0) = \underline{u}_k(t_0)$ and by Theorem 2.1, $\underline{u} \equiv \bar{u}$. This contradiction ends the proof.

2.5. THEOREM. *Let $F_i(t, x)$, $i = 1, ..., n$, be as above. If (2.1) holds, then the system (0.3) has a T-periodic solution $u^0 = (u_1^0, ..., u_n^0)$ such that $u_i^0 > 0$ for all i.*

Proof. Let $\bar{u}$, $\underline{u}$ be as above. By Theorem 2.1, we can assume that $n \geqslant 3$ and $\underline{u} < \bar{u}$.

Now, let us define $A = \{q \in \mathbb{R}^n : \underline{u}(0) \leqslant q \leqslant \bar{u}(0)\}$. By Proposition 2.4 we have that the Poincaré map $P: A \to A$; $P(q) = u(T, q)$ is well defined and the proof follows from Brouwer's fixed point theorem.

Note that Theorem 0.1 is a special case of Theorem 2.5.

3. UNIQUENESS AND STABILITY

In this section we study uniqueness and stability of the solution u^0 given by Theorem 0.1. To this end we shall assume that there are $c_1, ..., c_n > 0$ such that

$$\delta_i(t) := c_i b_{ii}(t) - \sum_{j \in J_i} c_j b_{ji}(t) \geqslant 0 \tag{3.1}$$

for all t in $\mathbb{R}$ and $i = 1, ..., n$.

3.1. LEMMA. *Let $C = (c_{ij})$ be a T-periodic complex $n \times n$ matrix function defined and continuous on $\mathbb{R}$ such that*

$$\alpha_i(t) := \mathrm{Re}(c_{ii}(t)) - \sum_{j \in J_i} |c_{ij}(t)| \geqslant 0, \qquad 1 \leqslant i \leqslant n. \tag{3.2}$$

Assume further that:

(a) *$c_{ij}(\tau) \neq 0$ if $j \in J_i$ and $\alpha_i(\tau) = 0$.*

(b) *$n > 2$ and $c_{ij}(t) > 0$ ($1 \leqslant i \leqslant n$; $t \in \mathbb{R}$) if $\sum_{i=1}^n \alpha_i \equiv 0$.*

If $\Phi(t)$ is the fundamental matrix, with $\Phi(0) =$ identity, of the linear differential system

$$x' = C(t)x, \tag{3.3}$$

then $|\lambda| > 1$ for each eigenvalue λ of $\Phi(T)$.

Proof. For $\varepsilon > 0$, let us define $C_\varepsilon(t) = C(t) + \varepsilon(\text{identity})$ and let $\Phi_\varepsilon(t)$ be the fundamental matrix, with $\Phi_\varepsilon(0) =$ identity, of the linear system $x' = C_\varepsilon(t)x$. Then, Theorem 2 of [6] implies that $\|\Phi_\varepsilon(t)x\|$ is an increasing function of t for each fixed x in $\mathbb{C}^n$. (Here, $\|x\| = \max\{|x_i| : 1 \leqslant i \leqslant n\}$ for all $x = (x_1, ..., x_n)$ in $\mathbb{C}^n$). Therefore, the same holds for $\|\Phi(t)x\|$.

Let λ be an eigenvalue of $\Phi(T)$ and fix z in $\mathbb{C}^n$, $\|z\|=1$, such that $\Phi(T)z=\lambda z$. Then, $|\lambda|\geqslant 1$, since $\|\Phi(T)z\|\geqslant\|\Phi(0)z\|$. Assume now that $|\lambda|=1$; then $\|\Phi(t)z\|$ is a constant function of t, since $\|\Phi(n)z\|=1$ for all integers n. So, $\|\Phi(t)z\|=1$.

Let us write $u(t)=(u_1(t), ..., u_n(t))=\Phi(t)z$ and fix τ in $\mathbb{R}$. Now, let us choose $i=1, ..., n$ such that $|u_i(\tau)|=1$. Then $|u_j(t)|\leqslant|u_i(\tau)|$ (for all t in $\mathbb{R}$ and $j=1, ..., n$) and by the arguments in [6] we get

$$\operatorname{Re}(c_{ii}(\tau))\leqslant\sum_{j\in J_i}|c_{ji}(\tau)|\,|u_j(\tau)|\leqslant\sum_{j\in J_i}|c_{ij}(\tau)|.$$

From this, $\alpha_i(\tau)=0$ and by the assumption (a), $|u_j(\tau)|=1$, for all j in J_i.

By the same argument, $\alpha_j(\tau)=0$, $j\in J_i$, and then $\alpha_k(\tau)=0$, for all τ in $\mathbb{R}$ and $k=1, ..., n$. So, by assumption (b) and (3.2), one has $n\geqslant 3$ and

$$c_{ii}(t)=\sum_{j\in J_i}c_{ij}(t),\qquad 1\leqslant i\leqslant n,\qquad t\in\mathbb{R}. \tag{3.4}$$

On the other hand, we have $|u_j(\tau)|=1$ for all τ in $\mathbb{R}$ and $j=1, ..., n$, and hence, $u_j\equiv\pm 1$. Without loss of generality, we can suppose that $u_1\equiv -1$, and by (3.3) we obtain $c_{11}=\sum_{j>1}c_{1j}u_j$. From this and (3.4), $u_j\equiv 1$, for all $j>1$.

Since $u_2\equiv 1$, we get

$$c_{21}=c_{22}+\sum_{j>2}c_{2j}=c_{21}+2\sum_{j>2}c_{2j},$$

and this contradiction ends the proof.

3.2. Corollary. *Assume* (0.2) *and* (3.1). *Then,* (0.1) *has exactly a positive and T-periodic solution* u^0. *Moreover,* u^0 *is locally asymptotically stable and, for* $n=2$,

$$u(t)-u^0(t)\to 0\qquad as\quad t\to\infty$$

for any positive solution u to (0.1).

Proof. Let $\bar{u}$, $\underline{u}$ be given by Theorem 2.1. According to part (c) of this theorem, we must prove that $\bar{u}=\underline{u}$. To do this, let us define

$$r(t)=\sum_{j=1}^{n}c_j\ln(\bar{u}_j(t)/\underline{u}_j(t)).$$

Then,

$$r'(t)=-\sum_{j=1}^{n}\delta_j(t)[\bar{u}_j(t)-\underline{u}_j(t)]\leqslant 0,$$

where δ_j is defined by (3.1). Hence, r is T-periodic and monotonic and so, $r' \equiv 0$.

Suppose now that $\bar{u} \neq \underline{u}$. By theorem 2.1(a) we have $\underline{u}(t) < \bar{u}(t)$ and therefore, $\delta_j \equiv 0$, $j = 1, ..., n$. That is,

$$c_i b_{ii} = \sum_{j \in J_i} c_j b_{ji}, \qquad i = 1, ..., n. \tag{3.5}$$

On the other hand, $\int_0^T a_i(t)\,dt = \int_0^T b_{ij}(t)\,U_j(t)\,dt$, since U_i is T-periodic, and by (0.2),

$$\int_0^T b_{ii}(t)\,U_i(t)\,dt > \sum_{j \in J_i} \int_0^T b_{ij}(t)\,U_j(t)\,dt, \qquad i = 1, ..., n.$$

Consequently,

$$\begin{aligned}\sum_{i=1}^n \int_0^T c_i b_{ii}(t)\,U_i(t)\,dt &> \sum_{i=1}^n \sum_{j \in J_i} \int_0^T c_i b_{ij}(t)\,U_j(t)\,dt \\ &= \sum_{i=1}^n \int_0^T \Bigl(\sum_{j \in J_i} c_j b_{ji}(t)\Bigr)\,U_i(t)\,dt.\end{aligned}$$

This contradicts (3.5) and the proof of our first assertion is complete.

Let us define $P\colon \mathbb{R}^n_+ := \{q \in \mathbb{R}^n : q \geqslant 0\} \to \mathbb{R}^n_+$, by $P(q) = u(T, q)$. We know that P has exactly a fixed point in $\operatorname{int}(\mathbb{R}^n_+)$, which we shall denote by q_0. By the arguments in Theorem 3.2 of [9], we have that λ is an eigenvalue of the Frechet derivative $P'(q_0)$ if and only if $1/\lambda$ is an eigenvalue of $\Phi(T)$, where $\Phi(t)$ is the fundamental matrix, with $\Phi(0) = \text{identity}$, of (3.3) with $c_{ij}(t) = c_j c_i^{-1} u_i(t, q_0)\, b_{ij}(t)$. From Lemma 3.1, we get $|1/\lambda| > 1$, if $n > 2$. Thus, $u^0(t) := u(t, q_0)$ is locally asymptotically stable.

Suppose now that $n = 2$ and define $p_0 = (0, 0)$, $p_1 = (U_1(0), 0)$, and $p_2 = (0, U_2(0))$. It is well known that $\{p_0, p_1, p_2\}$ is the set of all fixed points of P in the boundary of $\mathbb{R}^2_+$.

Given a fixed point q of P, we define

$$W^s(q) = \{p \in \mathbb{R}^2_+ : p^k(p) \to q \text{ as } k \to \infty\}.$$

It is well known that $W^s(p_0) = \{p_0\}$. Moreover, in Theorem 2.2 of [10], it is proved that $W^s(p_1) = (0, \infty) \times \{0\}$ and $W^s(p_2) = \{0\} \times (0, \infty)$. From Theorem 4.1 of [7] we have $W^s(q_0) = \operatorname{int}(\mathbb{R}^2_+)$ and the proof is complete.

Remark. Corollary 3.2 improves the main result in [9]. Compare also with Corollary 2.6 of [11].

4. An Improvement

In the following, C denotes the space of all bounded continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ and C_+ is the set of all $f \in C$ such that f is bounded below by a positive constant. Given f in C, we define the lower average of f by

$$A_L(f) = \lim_{r \to \infty} \inf_{t - s \geqslant r} (t-s)^{-1} \int_s^t f(\tau)\, d\tau. \tag{4.1}$$

Remarks. (a) For f in C we can define the upper average $A_M(F)$ of f by replacing in (4.1) inf by sup.

(b) If $f \in C$ is T-periodic then

$$A_L(f) = A_M(f) = T^{-1} \int_0^T f(t)\, dt.$$

(c) If $f \in C$ is almost periodic then

$$A_L(f) = A_M(f) = \lim_{T \to \infty} T^{-1} \int_0^{s+T} f(t)\, dt$$

uniformly on $s \in \mathbb{R}$. See [4, Chap. 3, Corollary 3.2].

It is not hard to prove the following result:

4.1. Theorem. *The problem*

$$x' = x[a(t) - b(t)x], \qquad x \in C_+,$$

has exactly one solution U if $a \in C$, $b \in C_+$, and $A_L(a) > 0$. In fact, this solution is given by

$$U(t) = \left[\int_{-\infty}^t b(s) \exp\left(-\int_s^t a(\tau)\, d\tau \right) ds \right]^{-1}.$$

Moreover, we have the following properties:

(a) U is T-periodic (almost periodic) if a, b are T-periodic (almost periodic).

(b) U is constant if a/b is constant. In this case, $U \equiv a/b$.

(c) $u(t) - U(t) \to 0$ as $t \to \infty$, for any positive solution to the equation $x' = x[a(t) - b(t)x]$.

(d) $\inf(a/b) \leqslant U \leqslant \sup(a/b)$.

4.2. PROPOSITION. *Assume that* $a_0, a_1 \in C$ *and* $b \in C_+$. *If* $U_i \in C_+$ *is a solution to the equation* $x' = x[a_i(t) - b(t)x]$, $i = 0, 1$, *and* $a_0 \leqslant a_1$, *then* $U_0 \leqslant U_1$.

Proof. Let us define $h = (U_0/U_1) - 1$. Then, $h' + bU_0 h \leqslant 0$. If $h(\tau) > 0$ for some τ, then $h'(\tau) < 0$ and, hence, $h > 0$ in $(-\infty, \tau]$. Consequently, $h' \leqslant -c$ in $(-\infty, \tau]$, for some constant $c > 0$. But $h \in C$ and this contradiction ends the proof.

Assume that $a_i, b_{ij} \in C_+$. We can apply our iterative scheme to prove the following improvement to Theorem 2.1.

4.3. THEOREM. *Let* U_i *be the unique solution to the problem*

$$x' = x[a_i(t) - b_{ii}(t)x], \qquad x \in C_+,$$

and suppose that

$$A_L\left(a_i - \sum_{j \in J_i} b_{ij} U_j\right) > 0, \qquad i = 1, \ldots, n. \tag{4.2}$$

Then, there exist $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$, $\underline{u} = (\underline{u}_1, \ldots, \underline{u}_n)$ *in* $C_+^n := C_+ \times \cdots \times C_+$ (n *times*) *such that* $\underline{u}_i \leqslant \bar{u}_i$ *and*

$$\begin{aligned} \bar{u}_i' &= \bar{u}_i\left[a_i - b_{ii}\bar{u}_i - \sum_{j \in J_i} b_{ij}\underline{u}_j\right] \\ \underline{u}_i' &= \underline{u}_i\left[a_i - b_{ii}\underline{u}_i - \sum_{j \in J_i} b_{ij}\bar{u}_j\right]. \end{aligned} \tag{4.3}$$

In particular, $(\bar{u}_1, \underline{u}_2)$, $(\underline{u}_1, \bar{u}_2)$ *are solutions to* (0.1) *if* $n = 2$. *Moreover,*

(1) $\bar{v} \leqslant \bar{u}$ *and* $\underline{u} \leqslant \underline{v}$, *for any solution* $(\bar{v}, \underline{v})$ *to* (4.3) *in* $C_+^n \times C_+^n$. *In particular,* $\underline{u} \leqslant u \leqslant \bar{u}$, *for any solution* u *to* (0.1) *in* C_+^n.

(2) $\bar{u} \equiv \underline{u}$ *if* $\bar{u}_k(\tau) = \underline{u}_k(\tau)$ *for some* (k, τ). *In this case,* $\bar{u}$ *is the unique solution to* (0.1) *in* C_+^n.

(3) $\bar{u}, \underline{u}$ *are constant if* b_{ij}/a_i *are constant for all* i, j.

4.4 COROLLARY. *If* (4.2) *holds, then the system* (0.1) *has one solution in* C_+^n.

Proof. If $n = 2$ or $\bar{u} \equiv \underline{u}$, there is nothing to prove, so we can assume that $n > 2$ and $\underline{u} < \bar{u}$.

For all integers $k \geqslant 1$, let us fix $p_k \in \mathbb{R}^n$ such that $\underline{u}(-k) < p_k < \bar{u}(-k)$, and let u^k be the solution to (0.1) determined by $u^k(-k) = p_k$. By the arguments in Proposition 2.4, we get

$$\underline{u}(t) < u^k(t) < \bar{u}(t) \qquad \text{for} \quad t > -k \text{ and } k = 1, 2, \ldots \tag{4.4}$$

In particular, $\{u^k(0)\}$ is a bounded sequence of $\mathbb{R}^n$, and we can assume, without loss of generality, that $\{u^k(0)\}$ converges to $p \in \mathbb{R}^n$. Now, let u be the solution to (0.1) given by $u(0) = p$; then, (4.4) implies $\underline{u}(t) \leqslant u(t) \leqslant \bar{u}(t)$ for all $t \in \text{domain}(u)$. From this, $\text{domain}(u) = \mathbb{R}$ and then u is the required solution. Thus, the proof is complete.

Uniqueness and "stability" of the solution u, given by Corollary 4.4, can be found in [11].

Conjecture. Corollary 4.4 remains true if we replace (4.2) by $A_M(a_i - \sum_{j \in J_i} b_{ij} U_j) < 0$, for $i = 1, \ldots, n$. This conjecture in proved in [7], in the periodic case, for $n = 2$.

5. A Particular Case

In this section we shall prove Theorem 0.2. Given a bounded function $f\colon (\tau, \infty) \to \mathbb{R}$, some $-\infty \leqslant \tau$, we define

$$f_{L\infty} = \liminf_{t \to \infty} f(t), \qquad f_{M\infty} = \limsup_{t \to \infty} f(t).$$

Assume further that f is differentiable. It is not too hard to prove that there exists a sequence (t_k) in (τ, ∞) such that $t_k \to \infty$, $h'(t_k) \to 0$, and $h(t_k) \to f_{L\infty}$ (resp. $f_{M\infty}$).

Proof of Theorem 0.2. Let $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$, $\underline{u} = (\underline{u}_1, \ldots, \underline{u}_n)$ be the solution to (4.3) given by Theorem 4.3. Since $\bar{u}$, $\underline{u}$ are constant, we get

$$1 = B_{ii}\bar{u}_i + \sum_{j \in J_i} B_{ij}\underline{u}_j \qquad \text{and} \qquad 1 = B_{ii}\underline{u}_i + \sum_{j \in J_i} B_{ij}\bar{u}_j,$$

where $B_{ij} := b_{ij}/a_i$. From this,

$$\theta_i = \sum_{j \in J_i} (B_{ij}/B_{jj})\, \theta_j, \tag{5.1$_i$}$$

where $\theta_i = B_{ii}(\bar{u}_i - \underline{u}_i)$. Note that (0.4) is equivalent to $\sum_{j \in J_i} B_{ij}/B_{jj} < 1$, for all i.

Now, let us fix $1 \leqslant k \leqslant n$ such that $\theta_k \geqslant \theta_i$, for all $i = 1, ..., n$. From $(5.1)_k$, we have $\theta_k = 0$ and then $\theta_i = 0$, for all $i = 1, ..., n$. From this, $\bar{u} = \underline{u}$ and, hence, $u^0 := \bar{u}$ is the unique solution to (0.1) in C^n_+.

Let $u = (u_1, ..., u_n)$ be a solution to (0.1) such that $u(t_0) > 0$ for some t_0. Then, u is defined and positive and bounded in $[t_0, \infty)$.

CLAIM. $u_{iL\infty} > 0$.

Proof. Let us fix $\varepsilon > 0$ such that

$$\sum_{j \in J_i} (B_{ij}/B_{jj}) + \varepsilon \sum_{j=1}^{n} B_{ij} < 1, \qquad \text{for} \quad i = 1, ..., n,$$

and let v_i be the solution to the initial value problem

$$x' = x[a_i(t) - b_{ii}(t)x], \qquad x(t_0) = u_i(t_0).$$

Then, $u_i(t) < v_i(t)$ for $t > t_0$.

On the other hand, $v_i(t) - U_i(t) \to 0$ as $t \to \infty$ and $U_i \equiv B_{ii}$. Consequently, there exists $t_1 > t_0$, such that

$$u_i(t) < \varepsilon + B_{ii}^{-1} \qquad \text{for} \quad t > t_i, i = 1, ..., n. \tag{5.2}$$

From the arguments in [9, Proposition 2.2], we have $u_i(t) \geqslant \min\{\varepsilon, u_i(t_1)\}$ for $t > t_1$, and the proof of the claim is complete.

Let us fix $i = 1, ..., n$, and a sequence $t_k \to \infty$, such that $u_i'(t_k) \to 0$ and $u_i(t_k) \to u_{iL\infty}$. From (0.1) and the above claim

$$1 \leqslant B_{ii} u_{iL\infty} + \sum_{j \in J_i} B_{ij} u_{jM\infty}. \tag{5.3}$$

Analogously,

$$1 \geqslant B_{ii} u_{iM\infty} + \sum_{j \in J} B_{ij} u_{jL\infty}. \tag{5.4}$$

Let us define $\theta_i = B_{ii}(u_{iM\infty} - u_{iL\infty})$. Then, (5.3)–(5.4) imply

$$\theta_i \leqslant \sum_{j \in J_i} (B_{ij}/B_{jj})\, \theta_j \qquad \text{for all} \quad i = 1, ..., n$$

and by the arguments above, $\theta_i = 0$ for all i. Moreover, we have equalities in (5.3)–(5.4) and, hence, $u_{iM\infty}$ $(= u_{iL\infty})$ is the ith coordinate of u^0. Thus, the proof is complete.

6. A Priori Bounds

Given $f \in C$, we define $f_M = \sup(f)$ and $f_L = \inf(f)$. We also define $B_{ij} = b_{ij}/a_i$, for all i, j, and we assume that

$$\sum_{j \in J_i} B_{ijM}/B_{jjL} < 1, \qquad 1 \leqslant i \leqslant n. \tag{6.1}$$

Note that (6.1) implies (4.2), since $B_{iiM}^{-1} \leqslant U_i \leqslant B_{iiL}^{-1}$.

From the iterative scheme

$$B_{iiL}\bar{\alpha}_{ik} = 1 - \sum_{j \in J_i} B_{ijL}\underline{\alpha}_{j,k-1}, \qquad B_{iiM}\underline{\alpha}_{ik} = 1 - \sum_{j \in J_i} B_{ijM}\bar{\alpha}_{jk}$$

with $\underline{\alpha}_{j0} = 0$, we get positive constants $\underline{\alpha}_i \leqslant \bar{\alpha}_i$, $1 \leqslant i \leqslant n$, such that

$$\begin{aligned} 1 &= B_{iiM}\underline{\alpha}_i + \sum_{j \in J_i} B_{ijM}\bar{\alpha}_j \\ 1 &= B_{iiL}\bar{\alpha}_i + \sum_{j \in J_i} B_{ijL}\underline{\alpha}_j. \end{aligned} \tag{6.2}$$

Actually, $\underline{\alpha}_i = \lim_{k \to \infty} \underline{\alpha}_{ik}$ and $\bar{\alpha}_i = \lim_{k \to \infty} \bar{\alpha}_{ik}$.

On the other hand, let $(\bar{u}_k)$, $(\underline{u}_k)$ be the sequences defined in Theorem 2.1. Then, we can prove (induction) that $\underline{\alpha}_{ik} \leqslant \underline{u}_{ik}$, $\bar{u}_{ik} \leqslant \bar{\alpha}_{ik}$, for all integers $k \geqslant 1$. For example, from Theorem 4.1(d), we have

$$\begin{aligned} \bar{u}_{i,k+1} &\leqslant \sup\left(b_{ii}^{-1}\left[a_i - \sum_{j \in J_i} b_{ij}\underline{u}_{jk}\right]\right) = \sup\left(B_{ii}^{-1}\left[1 - \sum_{j \in J_i} B_{ij}\underline{u}_{jk}\right]\right) \\ &\leqslant \left(1 - \sum_{j \in J_i} B_{ijL}\underline{\alpha}_{jk}\right)\Big/ B_{iiL} = \bar{\alpha}_{i,k+1}. \end{aligned}$$

Thus, we have proved the following result:

6.1. Theorem. *Assume* (6.1) *and let* $\bar{\alpha}_i$, $\underline{\alpha}_i$ *be given by* (6.1). *If* u *is a solution to* (0.1) *in* C_+^n, *then* $\underline{\alpha}_i \leqslant u_i \leqslant \bar{\alpha}_i$, *for all components* u_i *of* u.

This theorem improves some results in [1, 2, and 12].

Remark. For $n \geqslant 3$, we can prove that the results in this section remain true if we replace (6.1) by:

(a)$_i$ $\sum_{j \in J_i} B_{ijM}/B_{jjL} \leqslant 1$, for $i = 1, ..., n$.

(b) For some k, (a)$_k$ holds strictly.

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